Module 3

Topic: Mathematical Reasoning

The purpose of this professional development session is to highlight opportunities for teachers to develop students' reasoning, mathematical argumentation, and justification skills by determining when statements are always, sometimes, and never true. Students will become familiar seeking irregularities, testing conjectures, and finding examples and non-examples in an effort to increase conceptual understanding of topics in the six content standards for grades 9-12 defined by the Common Core State Standards for Mathematics.

Mathematical Elements

- Defining parameters for when a statement is always, sometimes, and never true.
- Find examples and non-examples.
- Defend reasoning with convincing argumentation.
- What counts as proof?
- Are examples and non-examples convincing justification?
- Emphasize importance of precise definitions.
- Opportunity to discuss concept of infinite for certain events that are always true.

Pedagogical Elements

- Mathematical Practice #3: Construct viable arguments and critique the reasoning of others.
- Boats, Dwyer, Laing, & Fratella (Mathematics Teaching in the Middle School, 2003): "Asking students to validate *always true* and *never true* conjectures can result in <u>a false sense of having examined all the cases</u>. The *sometimes true* statements are needed to help students dig deeper, gather evidence, test conjectures, and look for irregularities. When students engage in thinking and reasoning about all three types of conjectures, teachers can help them become appropriately cautious in making inferences and in examining all the conditions."
- We can help children develop the metacognitive skills that will serve in the study of mathematics by providing opportunities to analyze, justify, and refute conjectures.
- Providing and expecting use of definitions without opportunities to unpack what they really mean leave students with an incomplete understanding.
- Misconceptions can be formed by providing mathematically inaccurate universal statements.
- Exploration can lead to discovery of mathematical structures and the network of mathematical relationships.
- Students' responses to these kinds of statements can be used as a way to provoke classroom discussion.

Activities

Tasks

- Decide whether each of the following statements is always, sometimes, or never true. Justify your choice.

  Diagonals of a parallelogram are ______ perpendicular to one another.

  Diagonals of a trapezoid ______ bisect each other.

  Diagonals of a pentagon ______ have the same length.

  A figure with a larger perimeter ______ has a larger area.

  The medians of a triangle ______ divide its interior into 6 regions of equal area.

  Rectangles are ______ similar.

  A rotation followed by a rotation ______ results in a reflection.

- Discuss the Five Strands of Mathematical Proficiency (2001)
- Teachers select a few statements from the handout and discuss when they are always, sometimes, or never true. Each group selects one statement that provoked meaningful mathematical discussion and put statement and justifications on chart paper.
- Each group shares.
- If needed, share statements on PowerPoint:
  - When you add the same number to the numerator and the denominator of a fraction, the fraction becomes greater in value.
  - $x^2 < x^3$
  - Since 5 is less than 6, then one-fifth is less than one-sixth.
- Project-based questions from Jensen (2002, 2007)

Student Work Samples

- Data collected from the Year 2 Mathematics Coaching Program January virtual session demonstrates 5th and 6th grade students' reasoning to the following statement: Since 5 is less than 6, then one-fifth is less than one-sixth. (See handout)
- What can be learned from the students' responses?
- How might the responses be different if the task was given to students in grades 9-12?

Guided Questions for Debrief

- Where is the notion of developing reasoning abilities reflected in the Common Core?
- How can statements asking students provide justification be used as formative assessment?
- What questions could teachers ask students that extend the concept horizontally?
- What questions could teachers ask students that extend the concept vertically?
- What extension questions could be used?

Materials

Technology

- Laptop
- Document Projector
- Document Camera

Supplies

- Chart Paper
- Markers
- Copies of Always, Sometimes, and Never True handout
- Copies of Student Work Samples handout

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Strands of Mathematical Proficiency

National Research Council (2001)

- **Conceptual Understanding** – comprehension of mathematical concepts, operations, and relations
- **Procedural Fluency** – skill in carrying out procedures flexibly, accurately, efficiently, and appropriately
- **Strategic Competence** – ability to formulate, represent, and solve mathematical problems
- **Adaptive Reasoning** – capacity for logical thought, reflection, explanation, and justification
- **Productive Disposition** – habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy

Always, Sometimes, or Never True

Building Mathematical Reasoning

Select a few of the following statements for a topic you usually teach. Determine if each statement is *always true*, *sometimes true*, and *never true*. Provide a convincing justification for how and why you arrived at your conclusion.

- ### Number Theory

1. All operations are commutative.
2. The product of two nonprime numbers is a prime number.
3. A number can be both irrational and rational.
4. Dividing a whole number by a fraction yields a quotient that is greater than the whole number.
5. A matrix always has an inverse.
6. The identity matrix is commutative with all other matrices of the same square size.
7. If A, B, and c are matrices, then$A(B + C) = AB + CA$.
8. A geometric sequence grows faster than an arithmetic sequence.
9. If you add n consecutive numbers together, the result is divisible by n.
10. The sum of a rational number and an irrational number is irrational.
11. The product of a rational number and an irrational number is irrational.
12. If a whole number has an odd number of factors, then it is a perfect square.
13. If you add the same number to the top and bottom of a fraction, the fraction gets bigger in value.
14. A real number is also a complex number.
15. Infinity is a complex number.

- ### Algebra

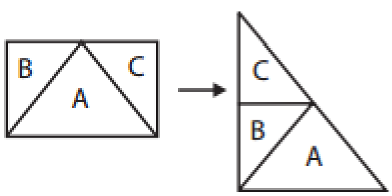
1. $x^2 = 2x$
2. $(x + y)^2 = x^2 + y^2$
3. $\sqrt{x^2} = x$
4. $x^2 < x^3$
5. If x is greater than y, and both are nonzero, then $\frac{1}{y} > \frac{1}{x}$.
6. If x is any even number, then x^2 is divisible by four.
7. $\sqrt{49} - \sqrt{49} = 0$
8. A system of two linear inequalities has a solution.
9. There are infinitely many polynomials with zeros a, b, and c.
10. The least common denominator of two rational expressions is the product of the denominators.

Functions

1. The domain of a square root function is the set of all non-negative real numbers.
2. The graph of $f(x) = 2^x$ lies in Quadrant I.
3. The functions $f(x)$ and $f(|x|)$have the same domain.
4. The functions $f(x)$ and $f(2x)$have the same range.
5. The inverse of a function is also a function.
6. The vertex of a parabola occurs at the minimum value of the function.
7. The graphs $f(x) = ax^2$ and $f(x) = -ax^2$ have the same width.
8. A quadratic function has two real solutions.
9. Composition of functions is commutative.
10. $f(x + y) = f(x) + f(y)$

Geometry

1. Right triangles can be equilateral.
2. Two triangles with the same perimeter also have the same area.
3. Two triangles with the same area also have the same perimeter.
4. If the side of a right triangle is 5 cm and another is 12 cm, then the third side must be 13 cm.
5. If a circle with diameter of length x and a square has side length x, then the area of the circle will be greater than the area of the square.
6. If a shape has an area of 9π, then the shape is a circle.
7. If the length of a right rectangular prism is doubled, then the surface area is also doubled.
8. If the volume of a right rectangular prims is doubled, then the surface area is also doubled.
9. When you cut a shape and rearrange the pieces, the area and perimeter stay the same.



10. When you cut a piece off a shape, you reduce its area and perimeter.
11. If the sides of a triangle are a, b, and c, then $a^2 + b^2 = c^2$.
12. If the midpoints of all the sides of an equilateral triangle are connected, and this process is repeated 3 more times with the resulting triangle from the previous step, then the area of the final triangle is $\frac{1}{256}$ the original area.
13. A line segment that is tangent to a circle O at its midpoint has its endpoints at A and B. If AB is the diameter of circle O, then triangle OAB is a right triangle.
14. If an angle inscribed in a circle is bisected, the bisection ray passes through the center.
15. In a triangle, the centroid and incenter are the same point.

Trigonometry

1. $\cos\theta = \cos(-\theta)$
2. $\sin\theta + \cos\theta = 1$
3. $\sin\theta = \cos\theta$
4. $4\sin^2\theta - 1 = 0$
5. $\sin\theta = -\sin(-\theta)$
6. $\sin\theta = \tan\theta$
7. $\sin(2x) = 2\sin(x)$
8. Doubling the amplitude of a trigonometric function doubles the period of the function.
9. Stretching the graph of a trigonometric function changes the period of the function.
10. Applying a phase shift of a secant graph changes the location of vertical asymptotes.

Statistics and Probability

1. The mean of a set of numbers is one of the numbers of that set.
2. The median of ten consecutive integers is one of those integers.
3. If the mode of a set of numbers is 14, then 14 is one of the numbers of that set.
4. The mean of a set of numbers is greater than the median of that set of numbers.
5. If you add a number to a set of numbers, the mean changes.
6. The probability of an event occurring is greater than 1.
7. The probability of an even occurring can be negative.
8. If two sets of numbers are combined, then the mean of the new set is the same as the mean of the 2 means.
9. If the size of a sample increases, then the standard deviation increases.
10. If the same value is added to each member of the set, then the mean doesn't change.
11. If the same value is added to each member of the set, then the median doesn't change.
12. If the same value is added to each member of the set, then the mode doesn't change.
13. If you get 85% on a test where the mean is 80%, and 65% of a different test where the mean is 60%; then the score of 65% is better than 85%.
14. Half of the students taking a test score less than the mean score.
15. $P(A\ or\ B) = P(A\ and\ B)$

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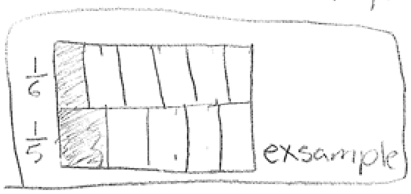
Student Work Samples

Five is less than six, so one-fifth is less than one-sixth.

Student #1

1. Five is less than six, so one-fifth is less than one-sixth.

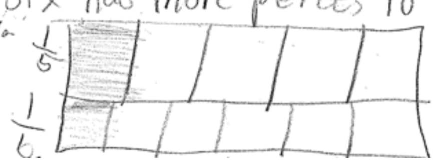
Never true, because the larger the denominator the smaller it gets like, $\frac{1}{10}$ and $\frac{1}{7}$ is greater than $\frac{1}{10}$.



Student #2

1. Five is less than six, so one-fifth is less than one-sixth.

Sometimes true because it depends on the size of the parts. If there the same then one-fifth is bigger because one-six has more peices to fil than one-fifth as the pictures shows below.



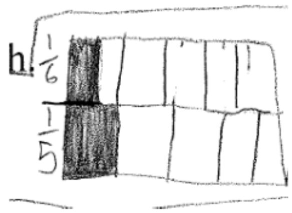
Student #3

1. Five is less than six, so one-fifth is less than one-sixth.

Never trere, because it takes less time to get to $\frac{1}{5}$ and it takes more to get to $\frac{1}{6}$.

Student #4

Never true beceuse, I have a picture and $\frac{1}{6}$ is smaller than $\frac{1}{5}$.



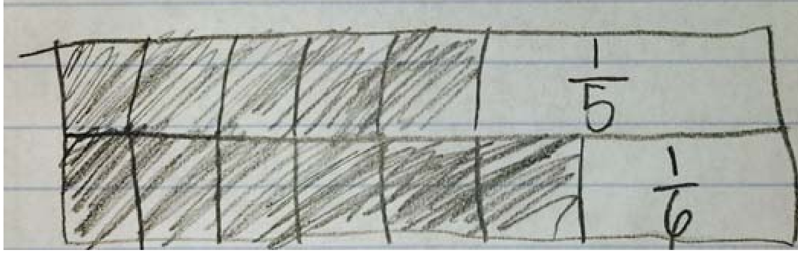
Student #5

1. Five is less than six, so one-fifth is less than one-sixth.

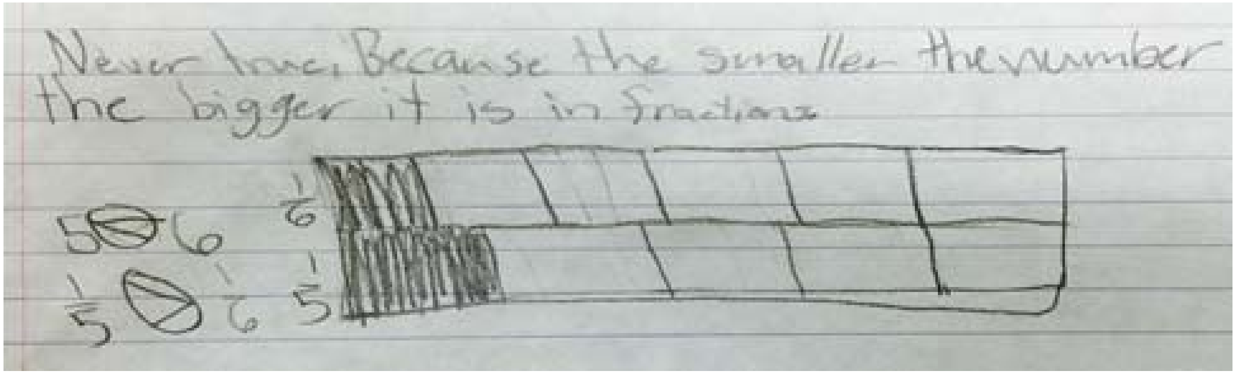
Sometimes true, because one-fifth might have a whole and one-sixth might not have a whole.

Student #6

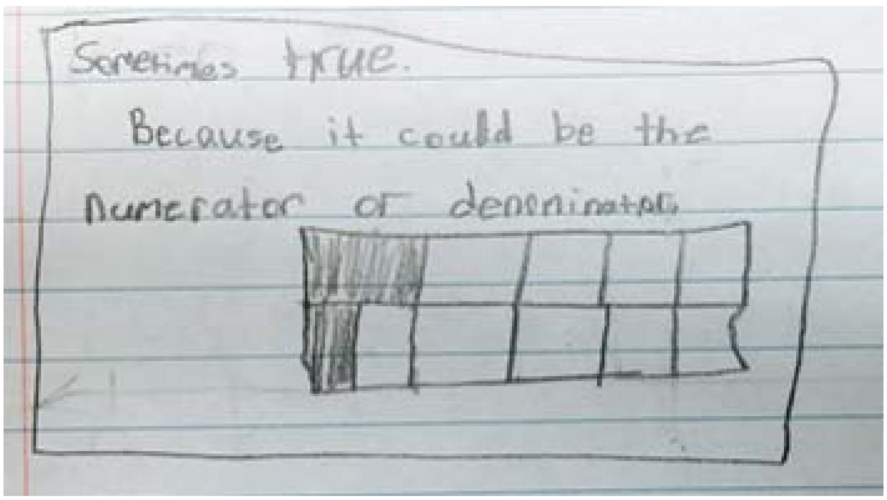
always true; because one of them one.six is Just six and six is bigger then one.fitn because it is Just fivo and six is bigger then five.



Student #7



Student #8



Student #9