The Ohio State University

Mathematical Reasoning

Module 3

Mathematics Teaching Institute, July 27-31, 2015

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Decide whether each of the following statements is always, sometimes, or never true. Justify your choice.

- Diagonals of a parallelogram are ________ perpendicular to one another.
- Diagonals of a trapezoid ________ bisect each other.
- Diagonals of a pentagon ________ have the same length.
- A figure with a larger perimeter ________ has a larger area.
- The medians of a triangle ________ divide its interior into 6 regions of equal area.
- Rectangles are ________ similar.
- A rotation followed by a rotation ________ results in a reflection.

What kind of thinking does the task motivate?

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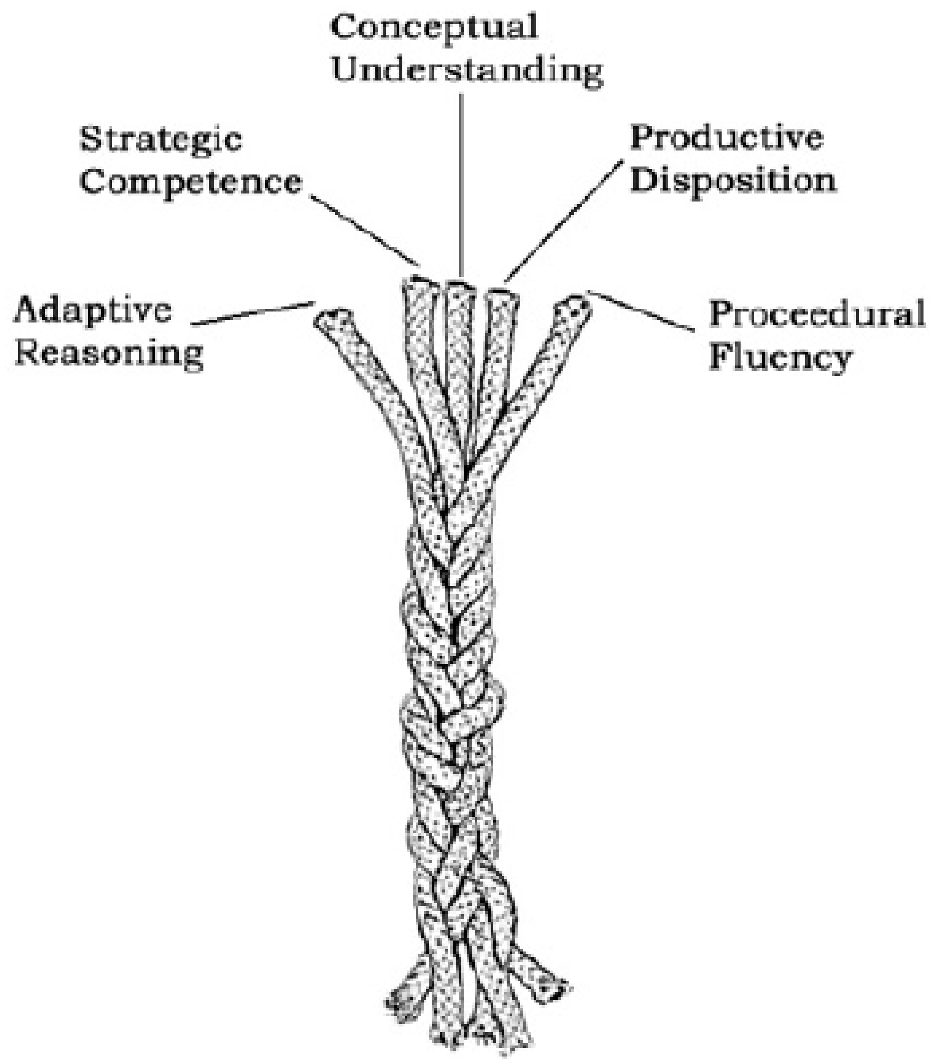
Five Strands of Mathematical Proficiency

(National Research Council, 2001)



- **Conceptual Understanding** – comprehension of mathematical concepts, operations, and relations
- **Procedural Fluency** – skill in carrying out procedures flexibly, accurately, efficiently, and appropriately
- **Strategic Competence** – ability to formulate, represent, and solve mathematical problems
- **Adaptive Reasoning** – capacity for logical thought, reflection, explanation, and justification
- **Productive Disposition** – habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy

Mathematical Practice #3:

Construct viable arguments and critique the reasoning of others.

- What counts as a viable argument?
- When is a critique sufficient?

- What does it mean to be always true?
- What does it mean to be sometimes true?
- What does it mean to be never true?

Boats, Dwyer, Laing, & Fratella
Geometric Conjectures: The Importance of Counterexamples
Mathematics Teaching in the Middle School, 2003

"Asking students to validate *always true* and *never true* conjectures can result in a **false sense of having examined all the cases**. The *sometimes true* statements are needed to help students dig deeper, gather evidence, test conjectures, and look for irregularities."

Determine if the statement is always, sometimes, or never true:

When you add the same number to the numerator and denominator of a fraction, the fraction becomes greater in value.

Example:

$$\frac{1}{2} < \frac{1+3}{2+3} \quad \rightarrow \quad \frac{1}{2} < \frac{4}{5}$$

Counterexample:

$$\frac{1}{2} < \frac{1+0}{2+0} \quad \rightarrow \quad \frac{1}{2} \not< \frac{1}{2}$$

Are examples and non-examples convincing enough?

Let's make some conjectures
about when it will be true,
and when it will be false?

We can show three cases $(b \neq 0)$:

$$\frac{a}{b} < \frac{a+n}{b+n} \qquad \frac{a}{b} > \frac{a+n}{b+n} \qquad \frac{a}{b} = \frac{a+n}{b+n}$$

Case 1:

$$\frac{a}{b} < \frac{a+n}{b+n} \qquad 1 < \frac{a+n}{b+n}\left(\frac{b}{a}\right) \qquad 1 < \frac{ab+bn}{ab+an}$$

When is $1 < \frac{ab+bn}{ab+an}$ true?

Case 2:

$$\frac{a}{b} > \frac{a+n}{b+n} \qquad 1 > \frac{a+n}{b+n}\left(\frac{b}{a}\right) \qquad 1 > \frac{ab+bn}{ab+an}$$

When is $1 > \frac{ab+bn}{ab+an}$ true?

Case 3:

$$\frac{a}{b} = \frac{a+n}{b+n} \qquad 1 = \frac{a+n}{b+n}\left(\frac{b}{a}\right) \qquad 1 = \frac{ab+bn}{ab+an}$$

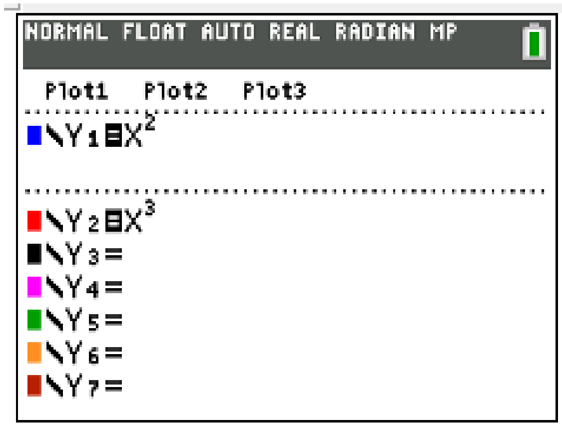
When is $1 = \dfrac{ab+bn}{ab+an}$ true?

Extensions:

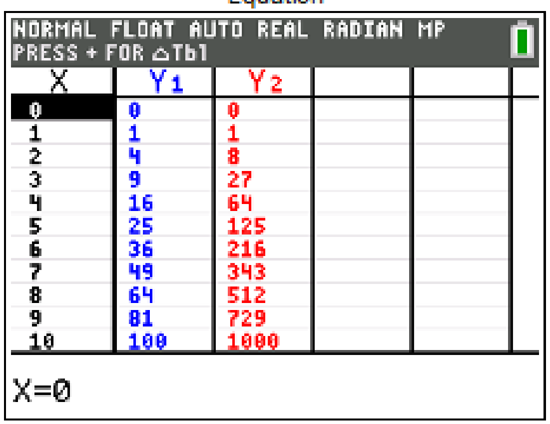
- What if now multiplying the same number?
- What if now subtracting the same number?
- What if now dividing the same number?
- Compare $\frac{a+n}{b+n}$ with $\frac{a}{b}+n$.

Determine if this statement is always, sometimes, or never true:

$$x^2 < x^3$$



Equation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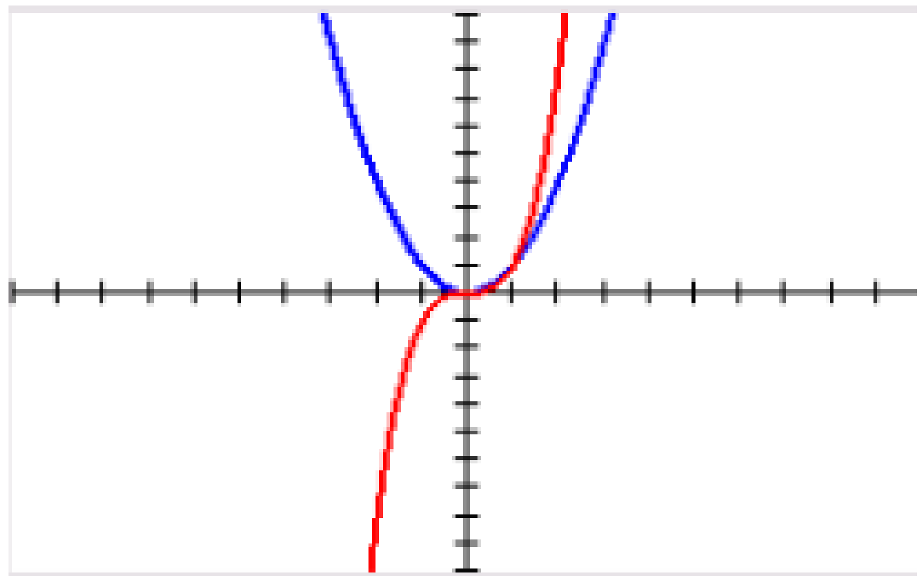
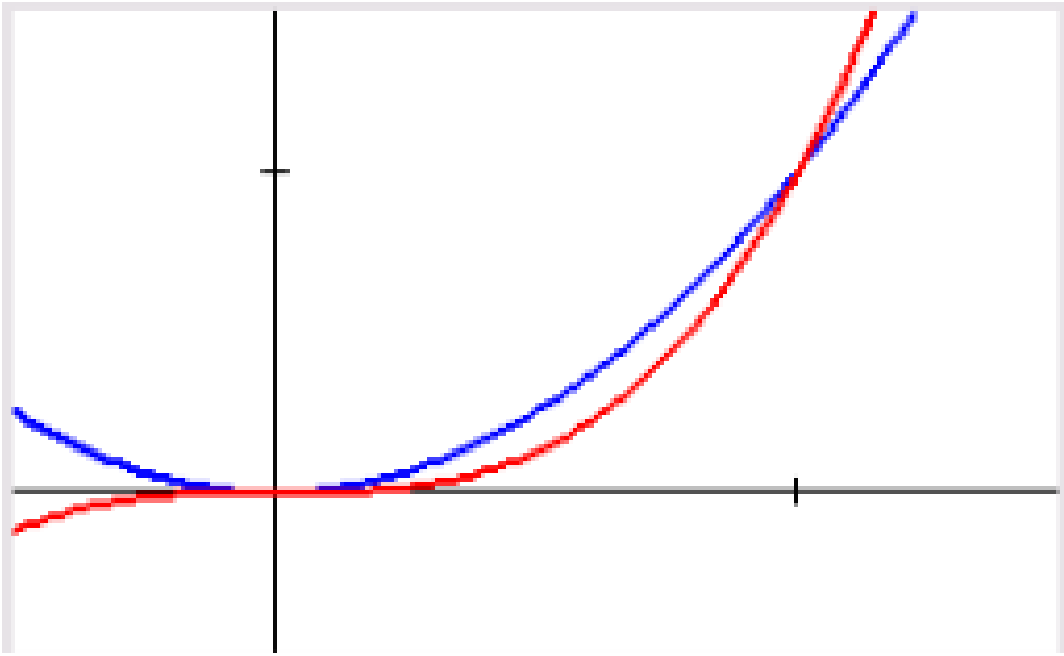
X	Y1	Y2
0	0	0
1	1	1
2	4	8
3	9	27
4	16	64
5	25	125
6	36	216
7	49	343
8	64	512
9	81	729
10	100	1000

X=0





Determine if this statement is always, sometimes, or never true:

Five is less than six, so one-fifth is less than one-sixth.

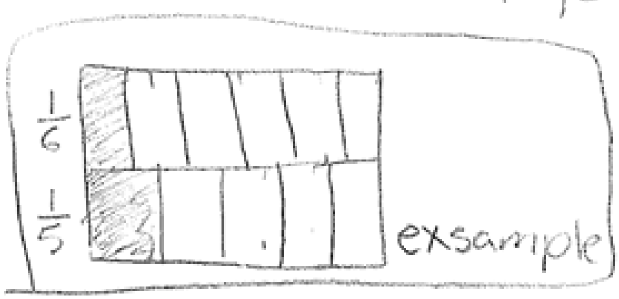
Consider the student work samples (5th & 6th grade)

- How are the ways these children reason different from the ways your students reason?
- What can we learn from these samples?

Student Work Sample #1

1. Five is less than six, so one-fifth is less than one-sixth.

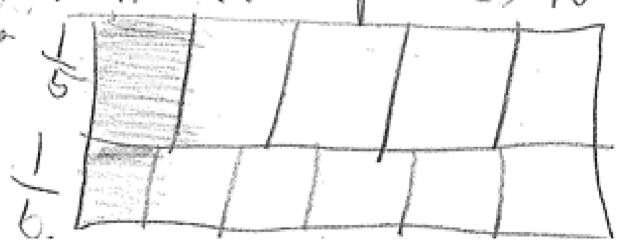
Never true, because the larger the denominator the smaller it gets like, $\frac{1}{10}$ and $\frac{1}{7}$ is greater than $\frac{1}{10}$.



Student Work Sample #2

1. Five is less than six, so one-fifth is less than one-sixth.

Sometimes true because it depends on the Size of the parts. If there the same then one-fifth is bigger because one-six has more peices to fil than one-fifth as the pictures shows below



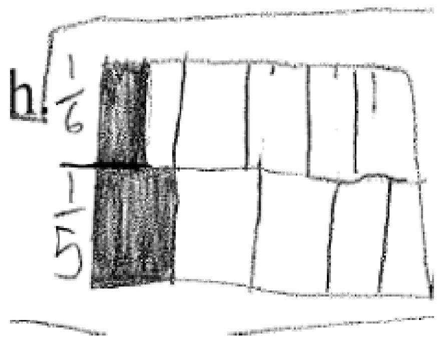
Student Work Sample #3

1. Five is less than six, so one-fifth is less than one-sixth.

Never true, because it takes less time to get to $\frac{1}{5}$ and it takes more to get to $\frac{1}{6}$.

Student Work Sample #4

Never true beceuse, I have a picture and $\frac{1}{6}$ is smaller than $\frac{1}{5}$.

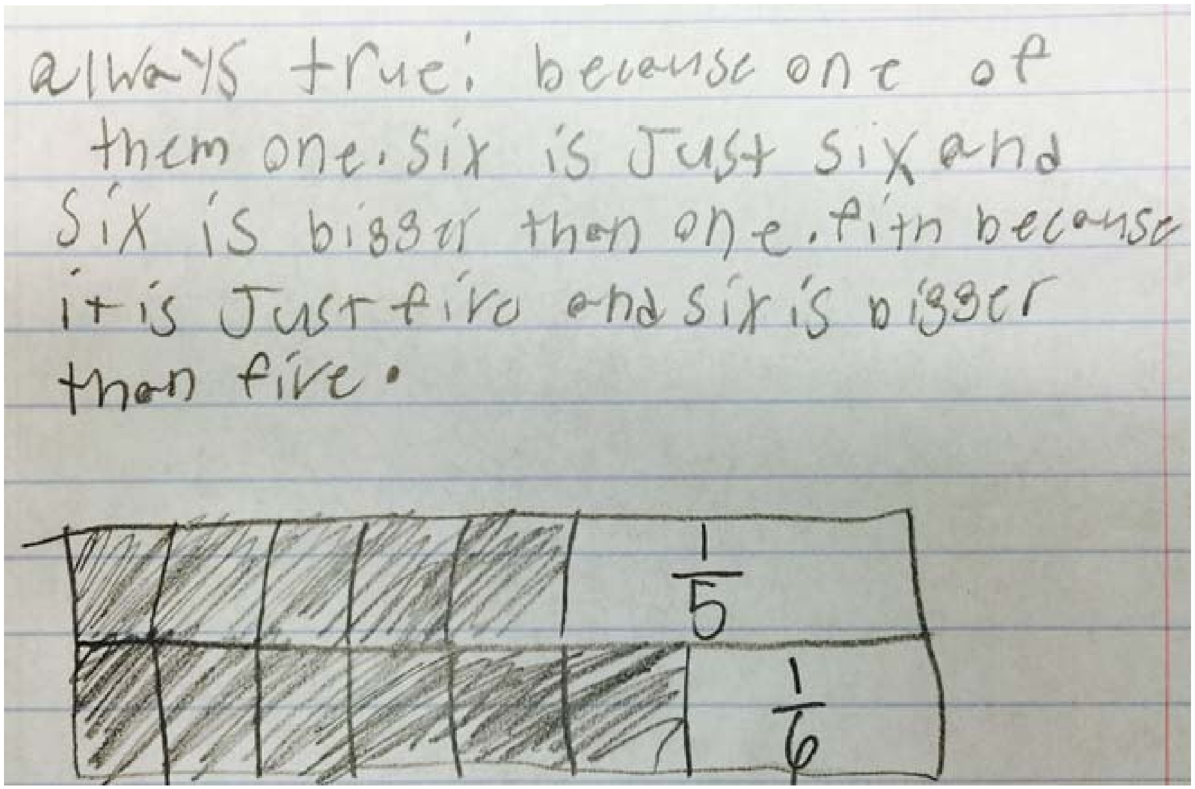


Student Work Sample #5

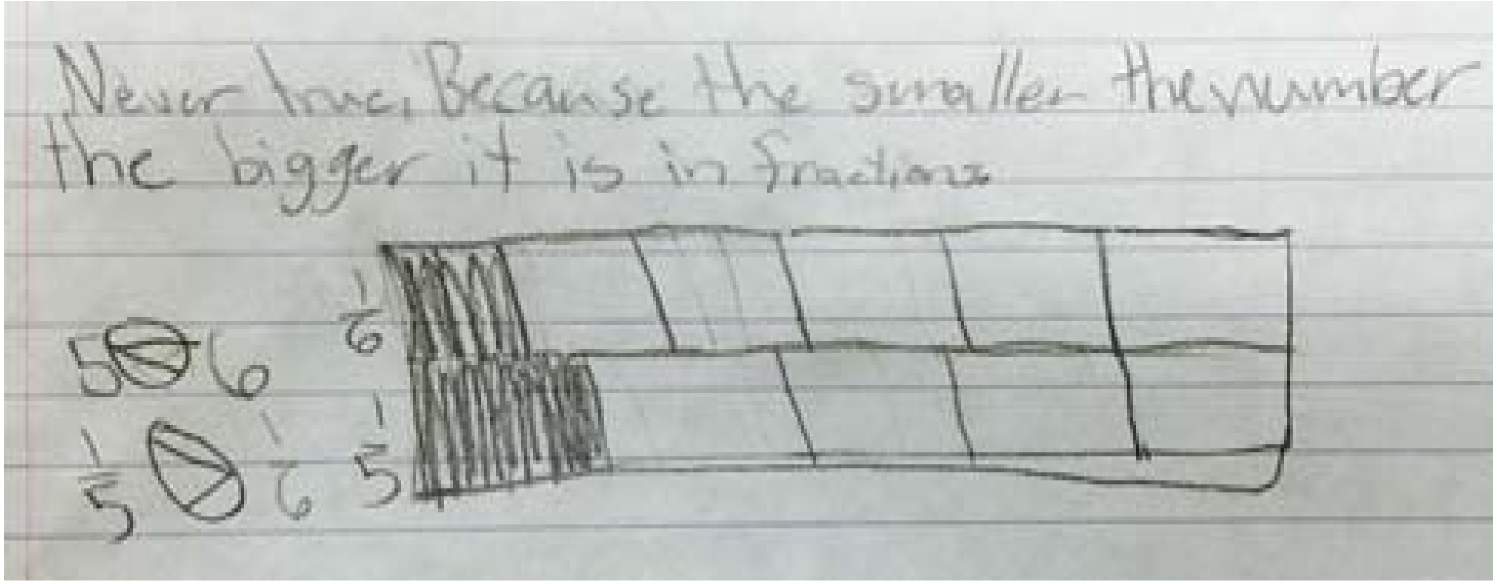
1. Five is less than six, so one-fifth is less than one-sixth.

Sometimes true, because one-fifth might have a whole and one-sixth might not have a whole,

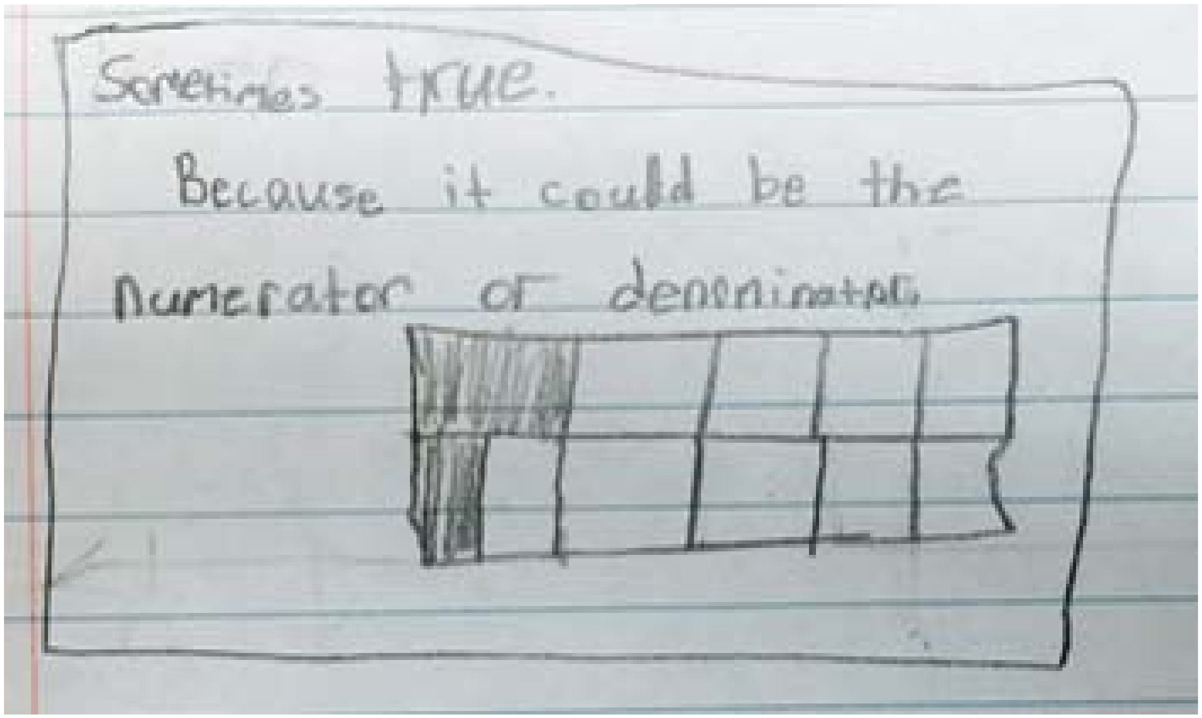
Student Work Sample #6



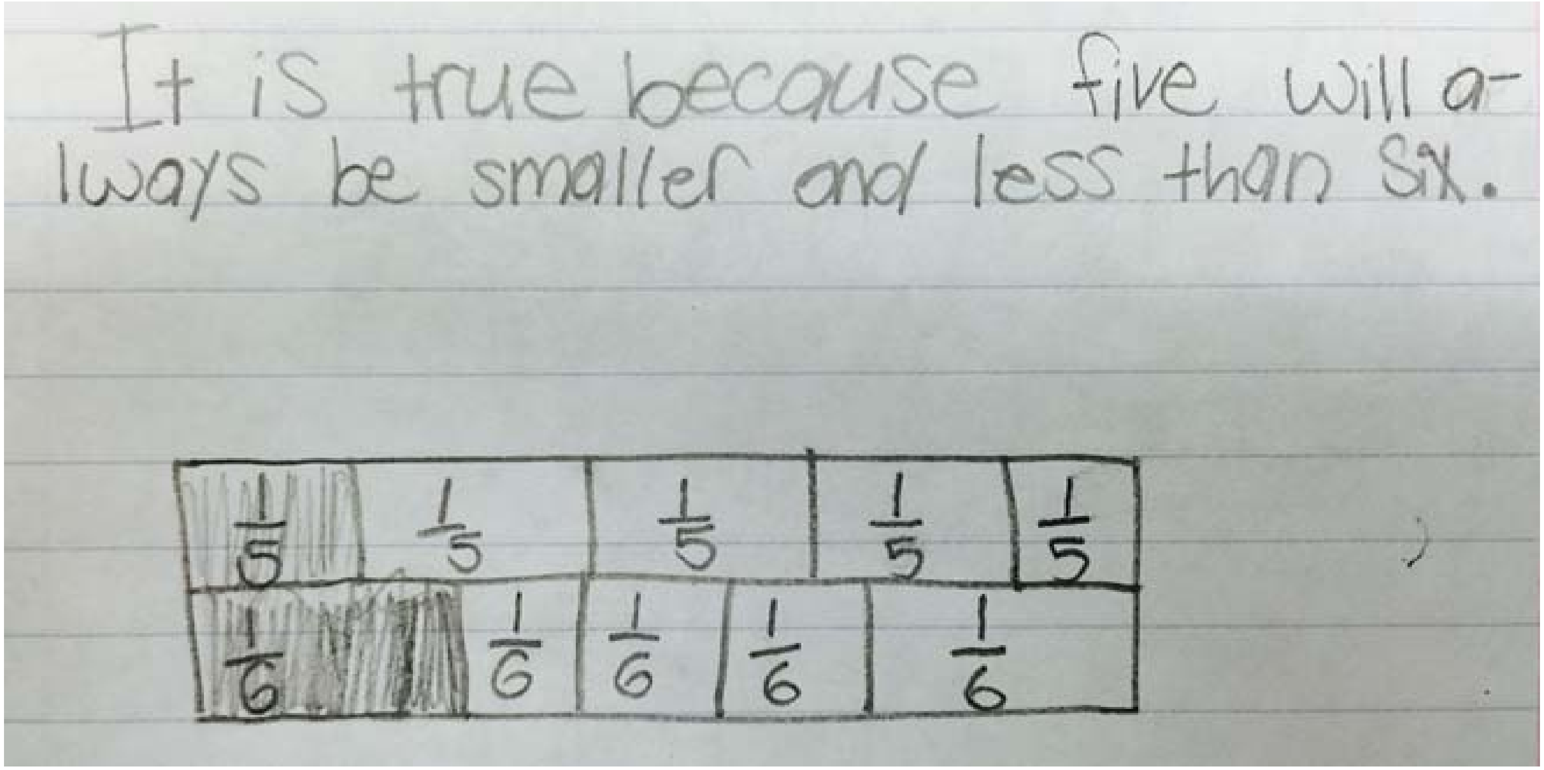
Student Work Sample #7



Student Work Sample #8



Student Work Sample #9



We can help children develop the metacognitive skills that will serve in the study of mathematics by providing opportunities to analyze, justify, and refute conjectures.

We can even use the students' responses to these kind of statements as a way to provoke classroom discussion!

Jensen (2002, 2007)

Consider the following tasks:

- What is the relation between one's income and the tax paid?
- What is the cost of me?
- Which means of transportation is the best?
- Can one become slim by exercising?
- How many windmills should Columbus have?
- What is the best shape of a can?
- How much fabric does one need to make a cloth for the dinner table?
- How many times can one brush one's teeth with a tube of toothpaste?
- Draw a sketch of a 135 m^2 house.
- How far away is the horizon?

Jensen (2002, 2007)

- How far ahead must the road be clear for you to make a safe overtaking?
- At what angle of incline does a tower fall?
- When you buy something, is it better to get a percentage of the price in discount before or after the tax has been added?
- Which savings account do you prefer: The one that pays 8% in annual interest or the one that pays A fixed amount of $110 in annual interest?
- A theater increases the ticket price by 30%, which causes the income from the sale of tickets to go up by 17%. By how many percentages has the size of the audience changed?
- Between three cities of the same size, where should the only high school in the area be?

Mathematical Competency

Readiness to act in response to a certain kind of mathematical challenge of a given situation (Blomhøj & Jensen, 2007).

Mathematical Literacy

An individuals capacity to identify and understand the role that mathematics play in the world, to make well-founded judgments and to use and engage with mathematics in ways that meet the needs of that individual's life as a constructive, concerned and reflective citizen" (PISA, 2003, cited in Jensen 2007).

Problem-Tackling Competency

This competence partly involves being able to put forward different kinds of mathematical problems, pure as well as applied, open as well as closed, and partly being able to solve such mathematical problems in their already formulated form, whether posed by oneself or by others, and, if necessary or desirable, in different ways (Niss and Jensen).

Mathematical Problem-Solving Competency

The readiness to solve different kinds of mathematical problems in their already formulated form (Jensen, 2007).