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Mathematical Reasoning

Mathematics Teaching Institute, July 27-31, 2015

Sponsored by Ohio Department of Education

Decide whether each of the following statements is always, sometimes, or never true. Justify your choice.

- Diagonals of a parallelogram are _____ perpendicular to one another.
- Diagonals of a trapezoid _____ bisect each other.
- Diagonals of a pentagon _____ have the same length.
- A figure with a larger perimeter _____ has a larger area.
- Rectangles are _____ similar.
- A rotation followed by a rotation _____ results in a reflection.

What kind of thinking does the task motivate?

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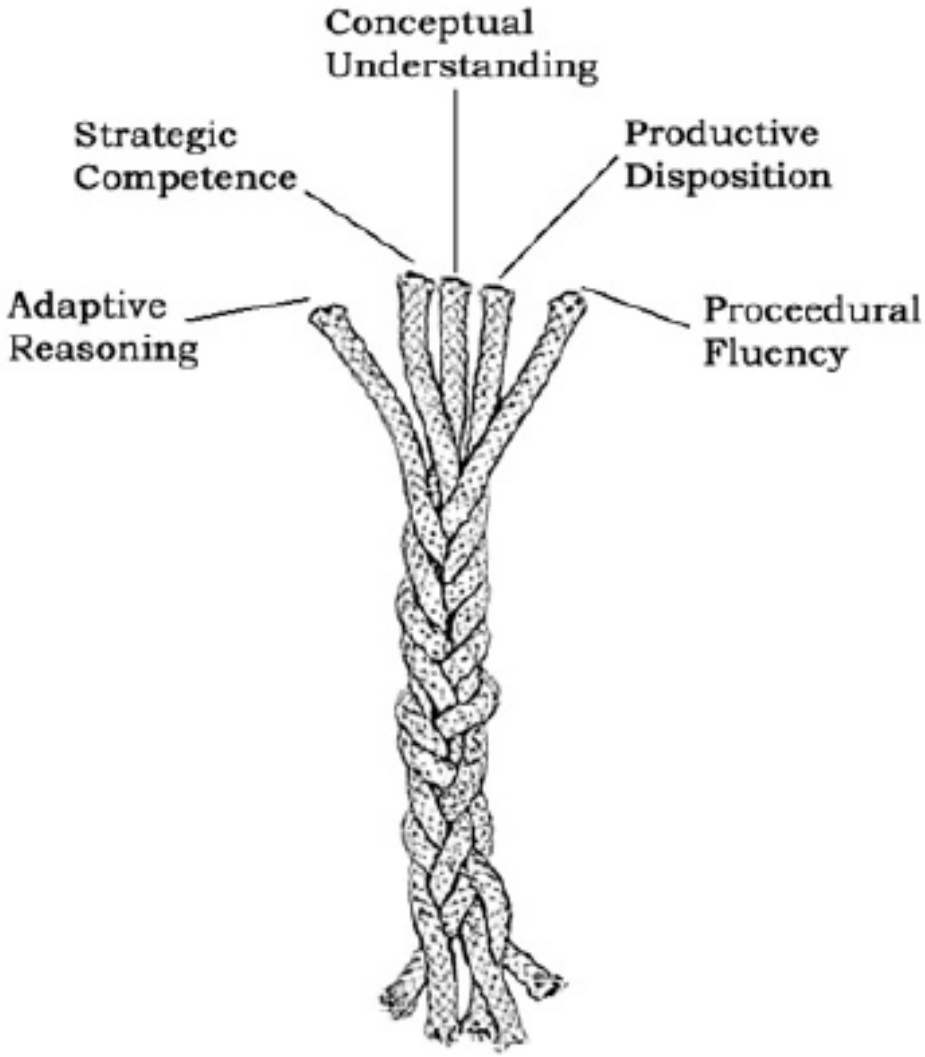
Five Strands of Mathematical Proficiency

(National Research Council, 2001)



- **Conceptual Understanding** – comprehension of mathematical concepts, operations, and relations
- **Procedural Fluency** – skill in carrying out procedures flexibly, accurately, efficiently, and appropriately
- **Strategic Competence** – ability to formulate, represent, and solve mathematical problems
- **Adaptive Reasoning** – capacity for logical thought, reflection, explanation, and justification
- **Productive Disposition** – habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy

Mathematical Practice #3:

Construct *viable arguments* and *critique* the reasoning of others.

- What counts as a viable argument?
- When is a critique sufficient?

Determine if the statement is always, sometimes, or never true:

When you add the same number to the numerator and denominator of a fraction, the fraction becomes greater in value.

- What does it mean to be *always* true?
- What does it mean to be *sometimes* true?
- What does it mean to be *never* true?

Example:

$$\frac{1}{2} < \frac{1+3}{2+3} \quad \rightarrow \quad \frac{1}{2} < \frac{4}{5}$$

Counterexample:

$$\frac{1}{2} < \frac{1+0}{2+0} \quad \rightarrow \quad \frac{1}{2} \not< \frac{1}{2}$$

Are examples and non-examples convincing enough?

We can show three cases ($b \neq 0$):

$$\frac{a}{b} < \frac{a+n}{b+n} \qquad \frac{a}{b} > \frac{a+n}{b+n} \qquad \frac{a}{b} = \frac{a+n}{b+n}$$

Case 1:

$$\frac{a}{b} < \frac{a+n}{b+n} \qquad 1 < \frac{a+n}{b+n}\left(\frac{b}{a}\right) \qquad 1 < \frac{ab+bn}{ab+an}$$

When is $1 < \frac{ab+bn}{ab+an}$ true?

Case 2:

$$\frac{a}{b} > \frac{a+n}{b+n} \qquad 1 > \frac{a+n}{b+n}\left(\frac{b}{a}\right) \qquad 1 > \frac{ab+bn}{ab+an}$$

When is $1 > \frac{ab+bn}{ab+an}$ true?

Case 3:

$$\frac{a}{b} = \frac{a+n}{b+n} \qquad 1 = \frac{a+n}{b+n}\left(\frac{b}{a}\right) \qquad 1 = \frac{ab+bn}{ab+an}$$

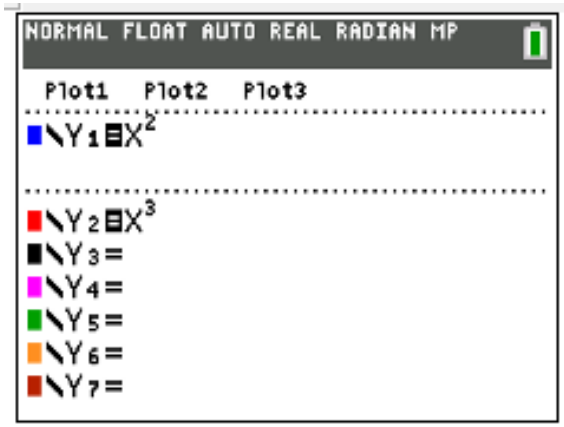
When is $1 = \frac{ab+bn}{ab+an}$ true?

Extensions:

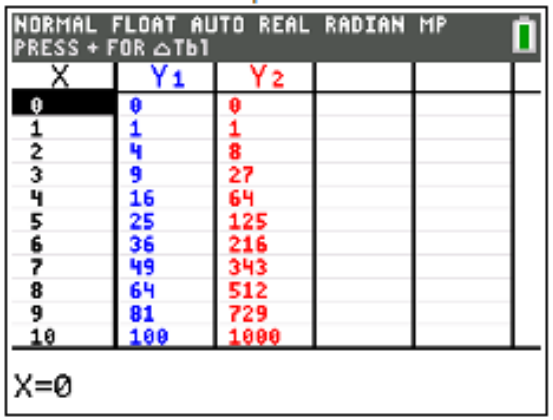
- What if now multiplying the same number?
- What if now subtracting the same number?
- What if now dividing the same number?
- Compare $\frac{a+n}{b+n}$ with $\frac{a}{b}+n$.

Determine if this statement is always, sometimes, or never true:

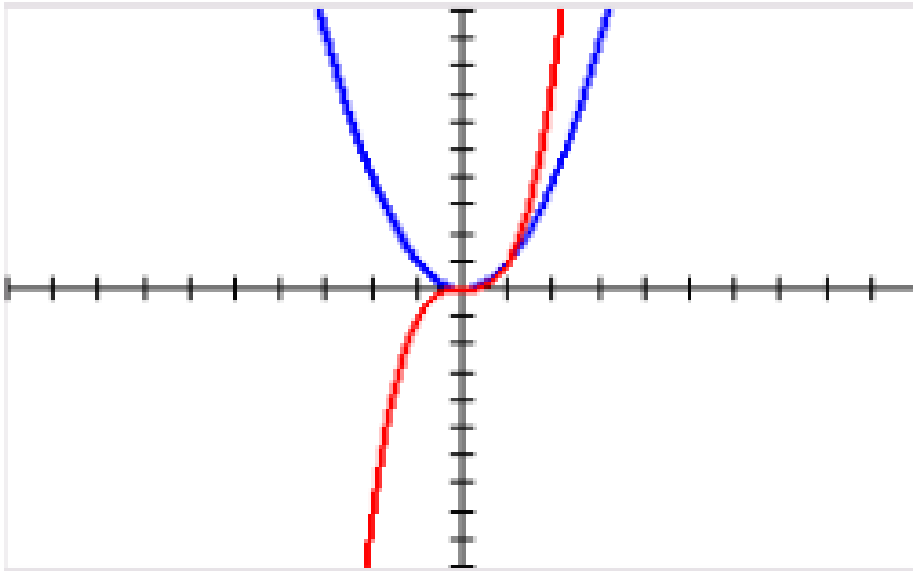
$$x^2 < x^3$$

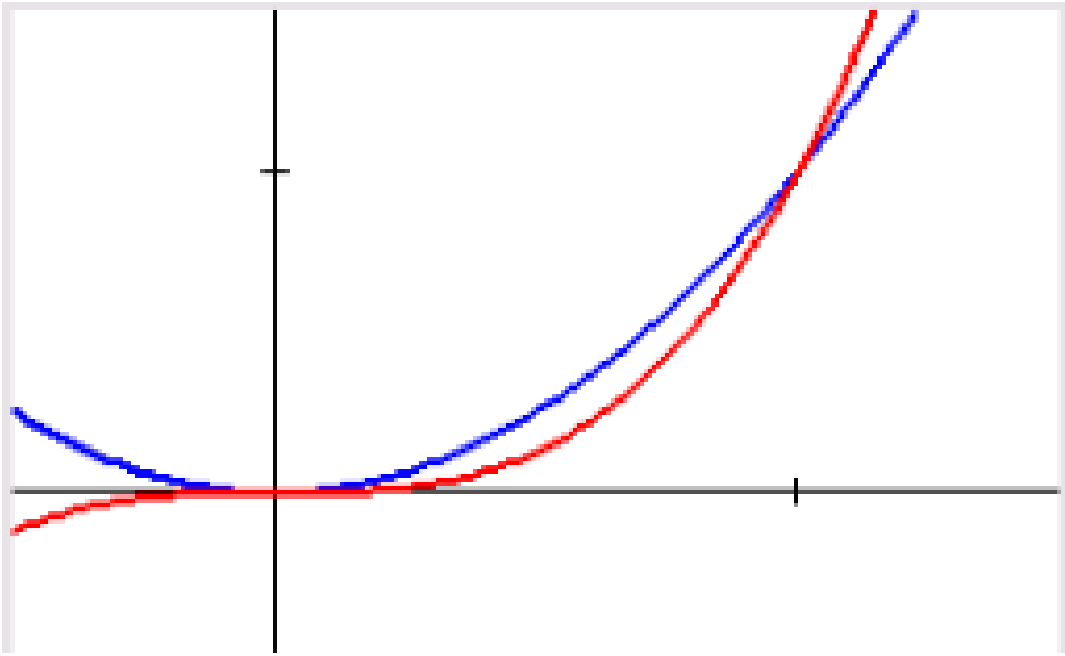


Equation



X	Y1	Y2
0	0	0
1	1	1
2	4	8
3	9	27
4	16	64
5	25	125
6	36	216
7	49	343
8	64	512
9	81	729
10	100	1000





Determine if this statement is always, sometimes, or never true:

Five is less than six, so one-fifth is less than one-sixth.

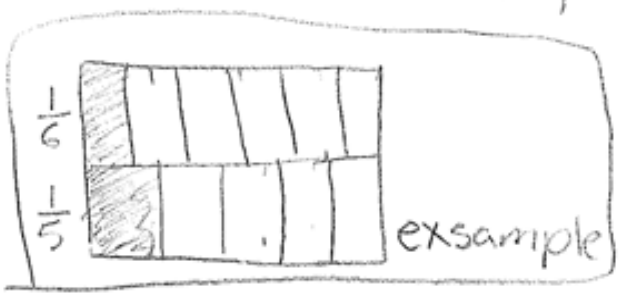
Consider the student work samples (5th & 6th grade)

- What can we learn from these samples?
- How are the ways these children reason different from the ways your students reason?
- How might the responses be different if the task was given to students in grades 7th and 8th students?

Student Work Sample #1

1. Five is less than six, so one-fifth is less than one-sixth.

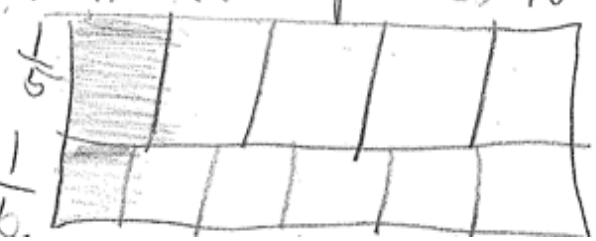
Never true, because the larger the denominator the smaller it gets like, $\frac{1}{10}$ and $\frac{1}{11}$ is greater than $\frac{1}{10}$.



Student Work Sample #2

1. Five is less than six, so one-fifth is less than one-sixth.

Sometimes true because it depends on the size of the parts. If there the same then one-fifth is bigger because one-six has more peices to fill than one-fifth as the pictures shows below.



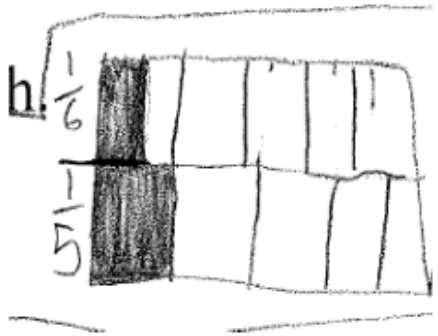
Student Work Sample #3

1. Five is less than six, so one-fifth is less than one-sixth.

Never true, because it takes less time to get to $\frac{1}{5}$ and it takes more to get to $\frac{1}{6}$.

Student Work Sample #4

Never true because, I have a picture and $\frac{1}{6}$ is smaller than $\frac{1}{5}$.

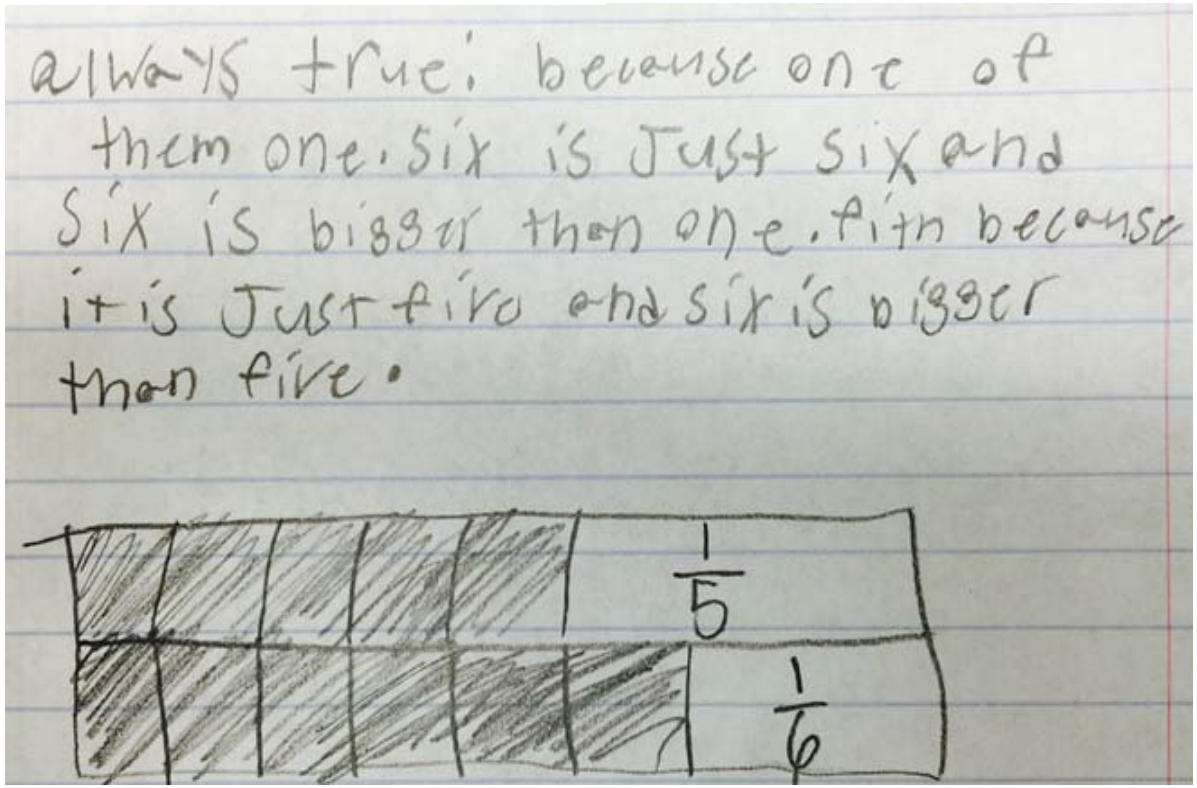


Student Work Sample #5

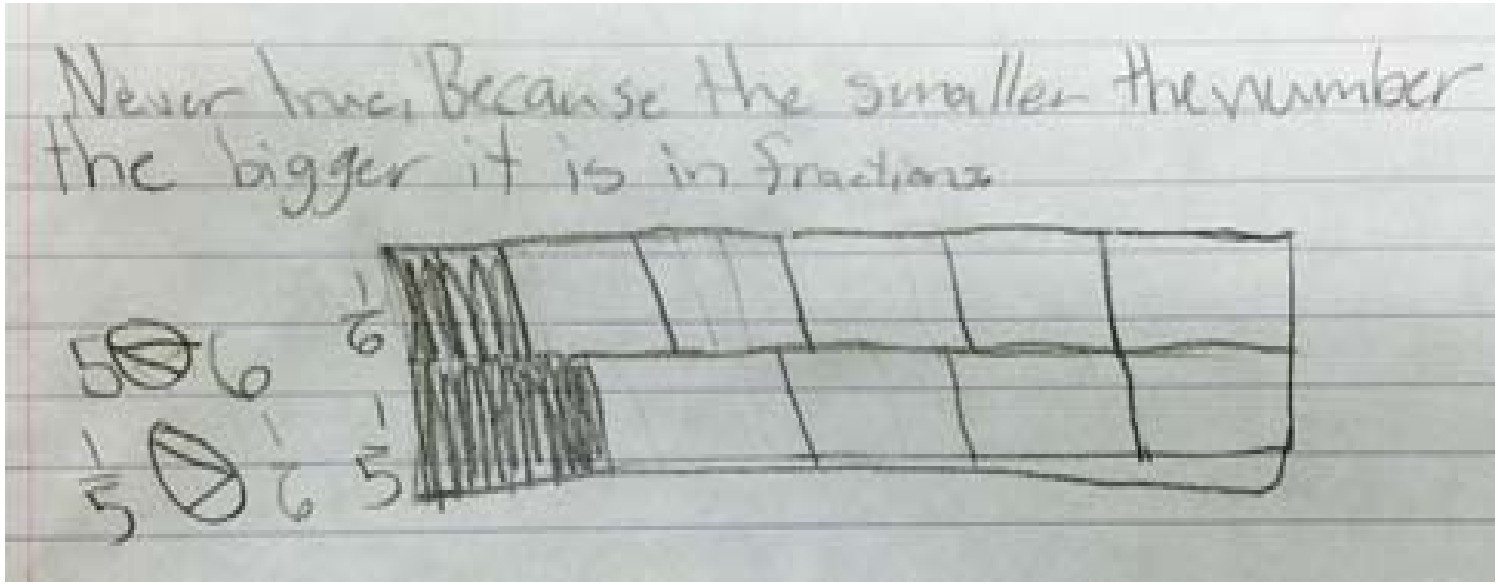
1. Five is less than six, so one-fifth is less than one-sixth.

Sometimes true, because one-fifth might have a whole and one-sixth might not have a whole,

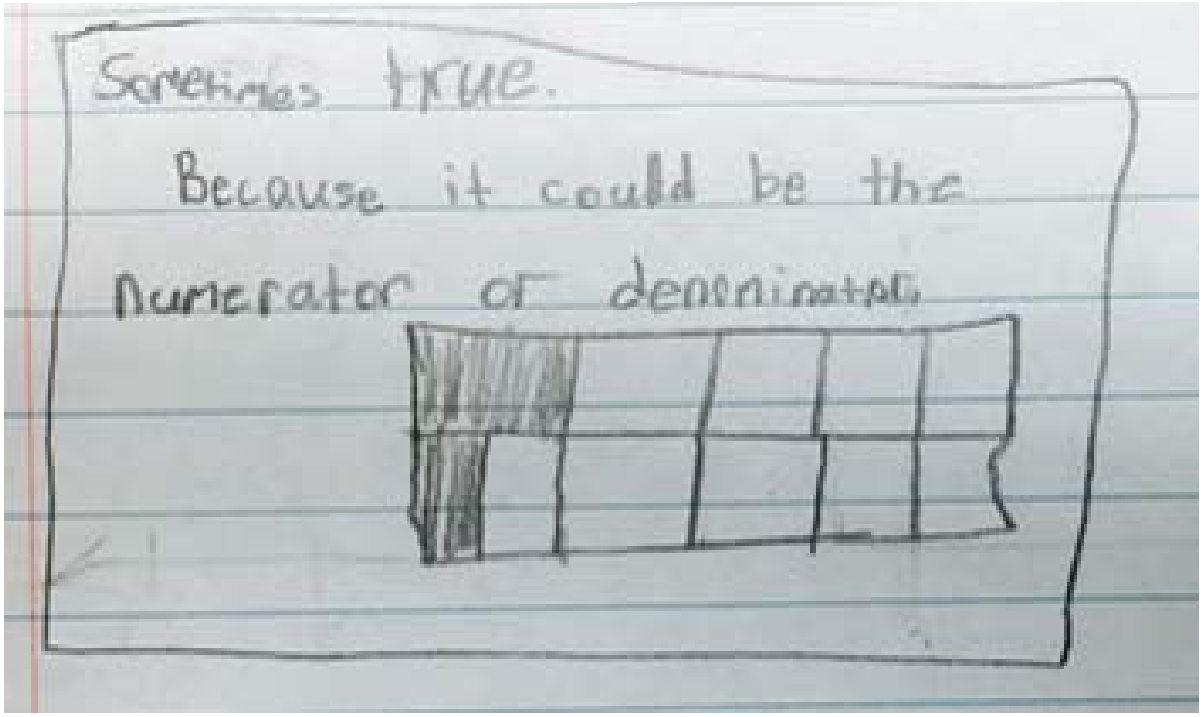
Student Work Sample #6


always true: because one of them one. six is just six and six is bigger than one. fifth because it is just five and six is bigger than five.

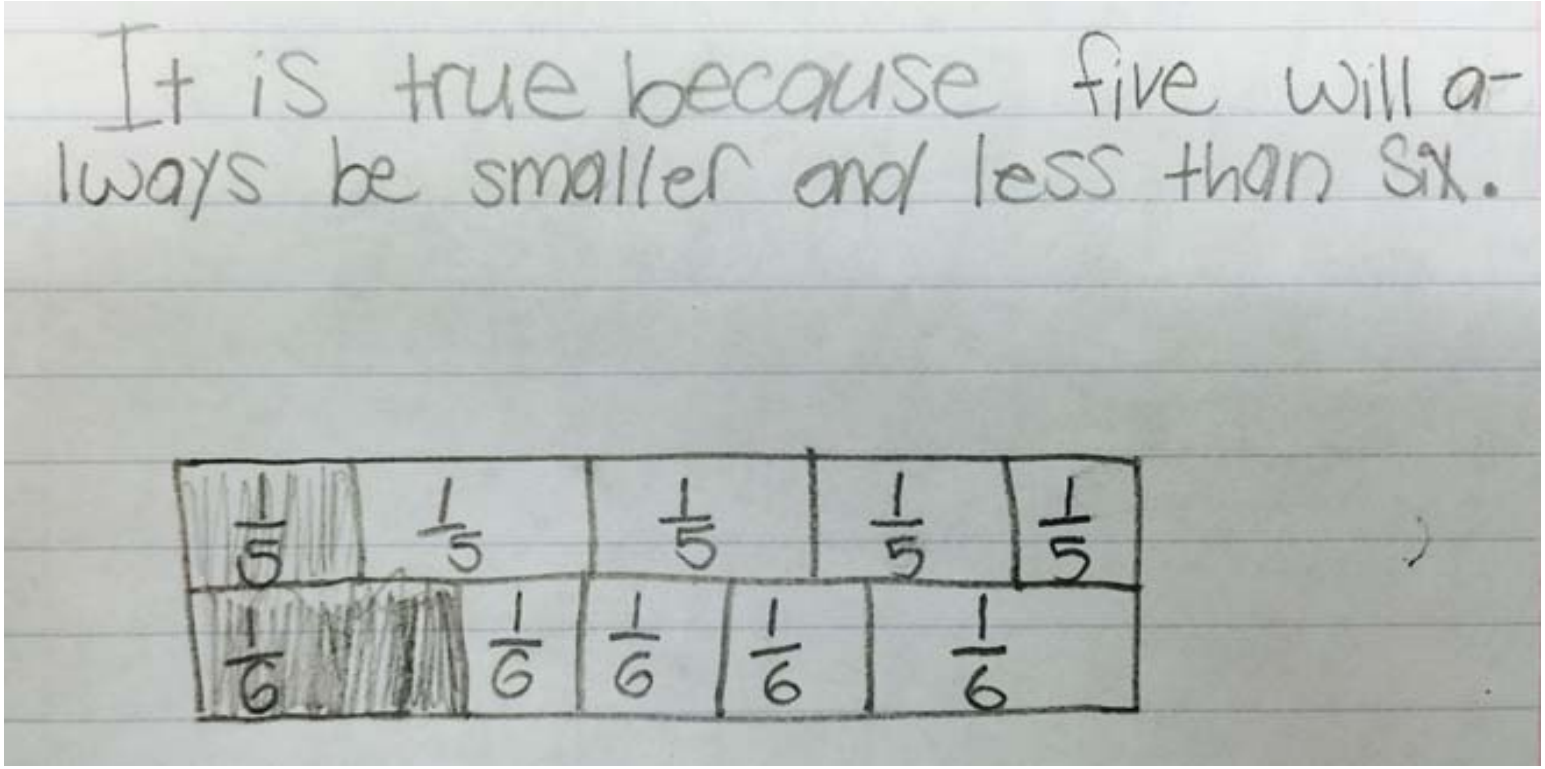
Student Work Sample #7



Student Work Sample #8



Student Work Sample #9



We can help children develop the skills that will serve in the study of mathematics by providing opportunities to analyze, justify, and refute conjectures.

We can even use the students' responses to these kind of statements as a way to provoke classroom discussion!

- Where is the notion of developing reasoning abilities reflected in the Common Core?
- How can statements asking students provide justification be used as formative assessment?
- What questions could teachers ask students that extend the concept horizontally?
- What questions could teachers ask students that extend the concept vertically?
- What extension questions could be used?

Building Mathematical Reasoning

Select a few of the following statements for a topic you usually teach. Determine if each statement is *always true*, *sometimes true*, and *never true*. Provide a convincing justification for how and why you arrived at your conclusion

Number Theory

1. All operations are commutative.
2. The product of two nonprime numbers is a prime number.
3. A number can be both irrational and rational.
4. Dividing a whole number by a fraction yields a quotient that is greater than the whole number.
5. A matrix always has an inverse.
6. The identity matrix is commutative with all other matrices of the same square size.
7. Infinity is a complex number.

… …

Algebra

1. $x^2 = 2x$
2. $(x+y)^2 = x^2 + y^2$
3. $\sqrt{x^2} = x$
4. $x^2 < x^3$
5. If x is greater than y, and both are nonzero, then $\frac{1}{y} > \frac{1}{x}$.
6. If x is any even number, then x^2 is divisible by four.
7. $\sqrt{49} - \sqrt{49} = 0$

… …

Functions

1. The domain of a square root function is the set of all non-negative real numbers.
2. The graph of $f(x) = 2^x$ lies in Quadrant I.
3. The functions $f(x)$ and $f(|x|)$have the same domain.
4. The functions $f(x)$ and $f(2x)$have the same range.
5. The inverse of a function is also a function.
6. The vertex of a parabola occurs at the minimum value of the function.

… …

Geometry

1. Right triangles can be equilateral.
2. Two triangles with the same perimeter also have the same area.
3. Two triangles with the same area also have the same perimeter.
4. If the side of a right triangle is 5 cm and another is 12 cm, then the third side must be 13 cm.
5. If a circle with diameter of length x and a square has side length x, then the area of the circle will be greater than the area of the square.

… …